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AN OPTIMAL CONDITION FOR THE LIL FOR TRIGONOMETRIC SERIES

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ABSTRACT. By a classical theorem (Salem-Zygmund [6], Erdős-Gál [3]), if (n_k) is a sequence of positive integers satisfying $n_{k+1}/n_k \geq q > 1$ $(k = 1, 2, \dots)$ then $(\cos n_k x)$ obeys the law of the iterated logarithm, i.e.,

$$\limsup_{N\to\infty} (N \log\log N)^{-1/2} \sum_{k\leq N} \cos n_k x = 1 \quad \text{a.e.} \tag{1}$$

It is also known (Takahashi [7, 8]) that the Hadamard gap condition $n_{k+1}/n_k \geq q > 1$ can be essentially weakened here but the problem of finding the precise gap condition for the LIL (1) has remained open. In this paper we find, using combinatorial methods, an optimal gap condition for the upper half of the LIL, i.e., the inequality ≤ 1 in (1).

1. INTRODUCTION

It is a well-known fact that lacunary subsequences of the trigonometric system behave like sequences of independent random variables. For example, if (n_k) is a sequence of positive integers satisfying the Hadamard gap condition

$$n_{k+1}/n_k \geq q > 1 \qquad (k = 1, 2, \dots) \tag{1.1}$$

then by classical results of Salem-Zygmund [5, 6] and Erdős-Gál [3] we have

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{2\pi}\lambda \left\{ 0 \leq x \leq 2\pi \colon \sum_{k\leq N} \cos n_k x < t\sqrt{N/2} \right\} \\ = (2\pi)^{-1/2} \int_{-\infty}^{t} e^{-u^2/2}\, du\end{aligned} \tag{1.2}$$

and

$$\limsup_{N\to\infty} (N \log\log N)^{-1/2} \sum_{k\leq N} \cos n_k x = 1 \quad \text{a.e.} \tag{1.3}$$

Actually, the exponential growth condition (1.1) can be weakened in the above results as the following theorem of Erdős [2] shows:

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Theorem A. *Let (n_k) be a sequence of positive integers satisfying*

$$n_{k+1}/n_k \geq 1 + c_k/\sqrt{k}, \qquad c_k \to \infty. \tag{1.4}$$

Then the sequence $(\cos n_k x)$ satisfies the central limit theorem (1.2). *On the other hand, for each $c > 0$ there exists a sequence (n_k) of positive integers such that*

$$n_{k+1}/n_k \geq 1 + c/\sqrt{k} \qquad (k \geq k_0)$$

but the central limit theorem (1.2) *is not valid.*

In terms of the growth speed of (n_k), Theorem A gives a precise criterion for $(\cos n_k x)$ to satisfy the central limit theorem. No similarly complete result exists in the case of the law of the iterated logarithm and in fact for subexponentially growing (n_k) the LIL turns out to be a much more delicate problem than the CLT. Takahashi proved (see [7, 8]) that the LIL (1.3) holds if (n_k) satisfies (1.4) with $c_k > k^\gamma$ for some $\gamma > 0$; in [1] we showed that this rate can be weakened to $c_k > (\log\log k)^\gamma$, $\gamma > \gamma_0$, and this result is essentially optimal in the sense that there exist sequences (n_k) satisfying (1.4) with $c_k = (\log\log k)^{1/2}$ such that the LIL (1.3) is false. The results of [1] show that for $c_k = (\log\log k)^\gamma$ the a.s. behavior of $(\cos n_k x)$ is very delicate: small changes of γ can lead to radical changes in the fluctuational properties of $(\cos n_k x)$, moreover, in the domain $c_k = (\log\log k)^\gamma$ the sequence $(\cos n_k x)$ exhibits highly unusual forms of "fractional" LIL behavior in the sense that it satisfies some forms of the LIL but fails similar, closely related LIL type results. The methods of [1] are not strong enough to determine the precise constants γ required for various forms of the LIL (such as the Kolmogorov-Erdős-Feller-Petrovski test, Chung type LIL's etc.); in fact even the optimal value of γ required for the ordinary LIL (1.3) has remained undetermined. The purpose of this paper is to improve the combinatorial tools of [1] and to find the precise gap condition for the upper half of the LIL. More precisely, we shall prove the following result.

Theorem. *Let (n_k) be a sequence of positive integers satisfying*

$$n_{k+1}/n_k \geq 1 + (\log\log k)^\alpha/\sqrt{k} \qquad (k \geq k_0) \tag{1.5}$$

for some $\alpha > 1/2$. Then $(\cos n_k x)$ satisfies

$$\limsup_{N\to\infty} (N \log\log N)^{-1/2} \left| \sum_{k\leq N} \cos n_k x \right| \leq 1 \quad a.e. \tag{1.6}$$

On the other hand, there exists a sequence (n_k) of positive integers satisfying (1.5) *with $\alpha = 1/2$ such that* (1.6) *is not valid.*

The second half of this theorem was proved in [1]; in fact, the example constructed in the proof of Theorem 2 of [1] shows that (1.5) with $\alpha = 1/2$ permits rather pathological LIL behavior of $(\cos n_k x)$: it can happen, e.g., that $(N\log\log N)^{-1/2}\sum_{k\leq N}\cos n_k x$ has a nonsymmetric cluster set, with its $\limsup < 1$ and $\liminf < -1$ a.e.

It seems likely that (1.5) with $\alpha > 1/2$ actually implies (1.6) with $\limsup = 1$ but this remains open. On the other hand, the results of [1] show that (1.5) with $\alpha > 5/2$ implies (1.6) with $\limsup = 1$.

It should be noted that the validity of the LIL (1.6) (or even of (1.3)) does not imply that the partial sum behavior of $(\cos n_k x)$ is exactly the same as that of independent r.v.'s. In [1] we constructed a sequence (n_k) of integers such that (1.2), (1.3) (and in fact (1.5) with $\alpha > 1/2$) hold and

$$\sum_{k \le N} \cos n_k x < (N \log\log N)^{1/2} \quad \text{a.e. for } N \ge N_0(x).$$

On the other hand, if X_k are independent r.v.'s with X_k distributed as $\cos n_k x$ (and hence the X_k are i.i.d) then by the classical Kolmogorov-Erdős-Feller-Petrovski test (see, e.g., [4]) we have for any increasing function $\varphi\colon (0, \infty) \to (0, \infty)$

$$P\left\{\sum_{k \le N} X_k < \sqrt{N/2}\,\varphi(N) \text{ a.s. for } N \ge N_0\right\} = 0 \text{ or } 1$$

according as

$$\sum_{N \ge 1} \frac{\varphi(N)}{N} \exp\left\{-\frac{1}{2}\varphi^2(N)\right\} < +\infty \text{ or } = +\infty.$$

In particular,

$$\sum_{k \le N} X_k \ge (N \log\log N)^{1/2} \quad \text{a.s. for infinitely many } N$$

and thus the partial sum behavior of $(\cos n_k x)$ is different from that of (X_k). By the main result of [1], $(\cos n_k x)$ satisfies the Kolmogorov-Erdős-Feller-Petrovski test if (1.5) holds with a sufficiently large α and thus the above pathological phenomenon cannot occur for large α. It is still open what is the smallest value of α implying the Kolmogorov-Erdős-Feller-Petrovski test for $(\cos n_k x)$; in [1] we showed that $\alpha > 5/2$ suffices while $\alpha \le 3/2$ does not. A comparison of the results of [1] and [4] suggests strongly that the critical value is $\alpha_0 = 3/2$ i.e., the Kolmogorov-Erdős-Feller-Petrovski test holds for $\alpha > 3/2$. (See in this respect also the remark at the end of our paper.)

The proof of our theorem will be combinatorial; in fact we shall deduce our result from estimates for the number of solutions of the diophantine equation

$$\pm n_{i_1} \pm n_{i_2} \pm \cdots \pm n_{i_p} = 0, \quad (1-\delta)N \le i_1, \ldots, i_p \le N.$$

This technique goes back (at least) to Erdős-Gál [3], but the subexponential domain presents considerable difficulties and we shall need essential improvement of earlier results to get the LIL (1.6) under the sharp gap condition (1.5), $\alpha > 1/2$.

2. A combinatorial lemma

The purpose of this section is to prove the following combinatorial statement which is the key step in the proof of our theorem.

Main Lemma. *Let $\{n_j, 1 \le j \le N\}$ be a finite sequence of positive integers satisfying*

$$n_{j+1}/n_j \ge 1 + c/\sqrt{j}, \qquad 1 \le j \le N-1. \tag{2.1}$$

Further let $p \geq 2$ *be an even integer and assume that*

$$N \geq c^2, \quad p \leq 4 \log\log N, \quad c \geq p^{1/2+\varepsilon} \tag{2.2}$$

for some $0 < \varepsilon \leq 1$. *Then for any integer* d *and* $0 < \delta \leq 1$ *the number of solutions of the equation*

$$\pm n_{i_1} \pm n_{i_2} \pm \cdots \pm n_{i_p} = d, \quad (1-\delta)N \leq i_1, \ldots, i_p \leq N, \tag{2.3}$$

is at most

$$\exp\left\{ \frac{Kp^{1-\varepsilon}\log^3 p}{\sqrt{\delta}} \right\} \frac{p!}{(p/2)!}(\delta N)^{p/2} \tag{2.4}$$

provided that $p \geq K$ *and* $N \geq N_0(\delta)$ *where* K *is an absolute constant.*

By the number of solutions of (2.3) we mean the number of $2p$-tuples $(i_1, \ldots, i_p, \varepsilon_1, \ldots, \varepsilon_p)$ where $i_1, \ldots, i_p$ are integers with $(1-\delta)N \leq i_1, \ldots, i_p \leq N$, the ε_j's are ± 1, and $\varepsilon_1 n_{i_1} + \cdots + \varepsilon_p n_{i_p} = d$. Call a solution $(n_{i_1}, \ldots, n_{i_p})$ of (2.3) *trivial* if $d = 0$ and among the terms in (2.3) each n_j occurs the same number of times with a positive as with a negative sign. It is easy to see that the number of trivial solutions of (2.3) is

$$(1 + o(1))\frac{p!}{(p/2)!}(\delta N)^{p/2}$$

as $N \to \infty$, uniformly in $2 \leq p \leq 4 \log\log N$. Hence our main lemma states that for any d the number of solutions of (2.3) exceeds the number of trivial solutions only by a subexponential factor $\exp(Cp^\gamma)$, $\gamma < 1$. As the example in §4 will show, for ε small enough, this subexponential factor cannot be removed (or improved beyond $O(p^{2-2\varepsilon})$) even for $d = 0$. Hence for ε small and p large, equation (2.3) with $d = 0$ can have many more nontrivial solutions than trivial ones. On the other hand, the proof of the main lemma will show that for $\varepsilon > 1$ the subexponential factor in (2.4) can be replaced by $1 + O(p^{-(\varepsilon-1)/6})$ or $O(p^{-(\varepsilon-1)/6})$ according as we include or exclude the trivial solutions. In other words, for $\varepsilon > 1$ and p large, most solutions of (2.3) with $d = 0$ are trivial. The remarkable consequences of this discrepancy will be discussed in §4.

We turn now to the proof of the main lemma. To simplify the formulas, we assume $\delta = 1$; the general case will require only trivial modifications. We break the argument into several steps.

Lemma 1. *Let* $\{n_j, 1 \leq j \leq N\}$ *be a sequence of positive integers satisfying* (2.1) *and assume* $N \geq c^2$. *Then for any* $0 < a < b$ *the interval* $[a, b]$ *contains at most*

$$2c^{-1}\sqrt{N}\log(b/a) + 1$$

terms of the sequence $\{n_j, 1 \leq j \leq N\}$.

Proof. Let n_q and n_r be the smallest and largest among the n_j's $(1 \leq j \leq N)$ in the interval $[a, b]$. Then $n_r/n_q \leq b/a$; on the other hand, by (2.1) we have

$$\frac{n_r}{n_q} \geq \prod_{j=q}^{r-1}\left(1 + \frac{c}{\sqrt{j}}\right) \geq \left(1 + \frac{c}{\sqrt{N}}\right)^{r-q} \geq \exp\left\{\frac{c}{2\sqrt{N}}(r-q)\right\}$$

using the fact that $1 + x \geq e^{x/2}$ for $0 \leq x \leq 1$. The two estimates for n_r/n_q imply

$$\exp\left\{\frac{c}{2\sqrt{N}}(r-q)\right\} \leq b/a$$

whence $r - q + 1 \leq 2c^{-1}\sqrt{N}\log(b/a) + 1$, as stated.

In what follows, fix an arbitrary sequence $\varepsilon_1, \ldots, \varepsilon_p$ of ± 1's and consider the equation

$$\varepsilon_1 n_{i_1} + \cdots + \varepsilon_p n_{i_p} = d, \qquad N \geq i_1 \geq i_2 \geq \cdots \geq i_p \geq 1. \tag{2.5}$$

In other words, in (2.3) we fix the signs and the order of the i_k's. To simplify the writing in the sequel, we introduce some terminology. Given a solution $(n_{i_1}, \ldots, n_{i_p})$ of (2.5), the ratios $n_{i_k}/n_{i_{k+1}}$ will be called the *gaps* in this solution. For any $1 \leq k \leq p-1$, define the positive integer j_k by $n_{i_k}/n_{i_{k+1}} \asymp 2^{j_k}$ where the symbol $a \asymp 2^j$ means $2^j \leq a < 2^{j+1}$. The gap $n_{i_k}/n_{i_{k+1}}$ is then called

$$\begin{array}{lll} & \text{small} & \text{if } p2^{-j_k} \geq 1/8, \\ (2.6) & \text{medium} & \text{if } c/(32\sqrt{N}) \leq p2^{-j_k} < 1/8, \\ & \text{large} & \text{if } p2^{-j_k} < c/(32/\sqrt{N}). \end{array}$$

In a solution $(n_{i_1}, \ldots, n_{i_p})$ of (2.5), the segment $(n_{i_k}, \ldots, n_{i_l})$ will be called a *block* if it contains no large gaps but it is preceded and followed by a large gap. (For $k = 1$ or $l = p$ there is only one side condition.) A block $(n_{i_k}, n_{i_{k+1}})$ of length 2 is called *trivial* if $i_k = i_{k+1}$ and $\varepsilon_k = -\varepsilon_{k+1}$; otherwise it is called *nontrivial*.

Lemma 2. *Let* $2 \leq k \leq p-1$ *and consider those solutions of* (2.5) *where* $n_{i_\nu}/n_{i_{\nu+1}} \asymp 2^{j_\nu}$ $(2 \leq \nu \leq k)$; *here* $j_2, \ldots, j_k$ *are fixed nonnegative integers. Then, given* $n_{i_1}, \ldots, n_{i_{k-1}}$, *the number of choices for* n_{i_k} *is at most*

$$\begin{cases} 3\sqrt{N}/c & \text{if } p2^{-j_k} \geq 1/8, \\ (48\sqrt{N}/c)\cdot p2^{-j_k} & \text{if } c/(32\sqrt{N}) \leq p2^{-j_k} < 1/8. \end{cases} \tag{2.7}$$

Note that the estimates in the first and second line of (2.7) are stated under the conditions that the gap $n_{i_k}/n_{i_{k+1}}$ is small or medium.

Proof. By $n_{i_{k-1}}/n_{i_k} \asymp 2^{j_{k-1}}$ we have $n_{i_k} \in [2^{-j_{k-1}-1}n_{i_{k-1}}, 2^{-j_{k-1}}n_{i_{k-1}}]$. Hence using Lemma 1 it follows that given $n_{i_1}, \ldots, n_{i_{k-1}}$, for n_{i_k} we have at most $(2\sqrt{N}/c)\log 2 + 1 \leq 3\sqrt{N}/c$ choices (provided $N \geq c^2$) no matter which assumption on $p2^{-j_k}$ in (2.7) holds. Assume now that $p2^{-j_k}$ satisfies the assumption in the second line of (2.7). Let $n_{i_1}, \ldots, n_{i_{k-1}}$ be given and let $\varepsilon_1 n_{i_1} + \cdots + \varepsilon_{k-1}n_{i_{k-1}} = A$. By $n_{i_k}/n_{i_{k+1}} \asymp 2^{j_k}$ it follows that the numbers $n_{i_{k+1}}, \ldots, n_{i_p}$ are all $\leq n_{i_k}\cdot 2^{-j_k}$ and thus

$$|\varepsilon_{k+1}n_{i_{k+1}} + \cdots + \varepsilon_p n_{i_p}| \leq p n_{i_k} 2^{-j_k}.$$

Hence (2.5) yields

$$A + \varepsilon_k n_{i_k}(1 + \theta p 2^{-j_k}) = d, \qquad |\theta| \leq 1.$$

Thus, setting $B = (d - A)/\varepsilon_k$ and using $p2^{-j_k} < 1/8$ and the fact that for $|x| \le 1/2$ we have $(1+x)^{-1} = 1 + \lambda x$ with $|\lambda| \le 2$, we get

$$n_{i_k} = B(1 + \theta p 2^{-j_k})^{-1} = B(1 + \theta' p 2^{-j_k}), \qquad |\theta'| \le 2.$$

Here $B \ne 0$ since $n_{i_k} \ne 0$. Thus using Lemma 1 and the assumption in the second line of (2.7) it follows that there are at most

$$(2\sqrt{N}/c) \cdot \log \frac{1 + 2p2^{-j_k}}{1 - 2p2^{-j_k}} + 1 \le (2\sqrt{N}/c) \cdot \log(1 + 8p2^{-j_k}) + 1$$
$$\le (16\sqrt{N}/c) \cdot p2^{-j_k} + 1 \le (48\sqrt{N}/c) \cdot p2^{-j_k}$$

choices for n_{i_k}.

Lemma 3. *Let* $1 \le k \le p-1$ *and consider those solutions of* (2.5) *where the gap* $n_{i_k}/n_{i_{k+1}}$ *is large. Then given* $n_{i_1}, \dots, n_{i_{k-1}}$ *(assuming nothing if* $k = 1$*) there is at most one possibility for* n_{i_k}.

This follows similarly as the second estimate in (2.7) in the previous lemma. Since the gap $n_{i_k}/n_{i_{k+1}}$ is large, we have $n_{i_k}/n_{i_{k+1}} \ge 32p\sqrt{N}/c$ and thus the numbers $n_{i_{k+1}}, \dots, n_{i_p}$ are all less than $n_{i_k} \cdot c/(32p\sqrt{N})$. Hence with $n_{i_1}, \dots, n_{i_{k-1}}$ given and setting $B = (d - A)/\varepsilon_k$, it follows as above that

$$n_{i_k} \in [B(1 - 2c/(32\sqrt{N})), B(1 + 2c/(32\sqrt{N}))]$$

and by Lemma 1 the above interval contains at most

$$\frac{2\sqrt{N}}{c} \log\left(1 + \frac{8c}{32\sqrt{N}}\right) + 1 \le 3/2$$

integers.

Lemma 4. *Fix* $k \ge 0$, $s \ge 1$ *and consider those solutions of* (2.5) *where* $(n_{i_{k+1}}, \dots, n_{i_{k+s}})$ *is a block. Then given* $(n_{i_1}, \dots, n_{i_k})$ *(assuming nothing if* $k = 0$*) the number of choices for the* s*-tuple* $(n_{i_{k+1}}, \dots, n_{i_{k+s}})$ *is at most*

$$\begin{array}{ll} \dfrac{(600 \log p)^s}{c^{s-2}} N^{s/2} & \text{if } s \ge 3, \\ 1 & \text{if } s = 1, \\ 3\sqrt{N}\log N & \text{if } s = 2 \text{ and the block } (n_{i_{k+1}}, n_{i_{k+2}}) \text{ is nontrivial}, \\ N & \text{if } s = 2 \text{ and the block } (n_{i_{k+1}}, n_{i_{k+2}}) \text{ is trivial}. \end{array} \tag{2.8}$$

Proof. With $n_{i_1}, \dots, n_{i_k}$ given, set $A = \varepsilon_1 n_{i_1} + \dots + \varepsilon_k n_{i_k}$. Then the remaining terms $n_{i_{k+1}}, \dots, n_{i_p}$ satisfy the equation

$$\varepsilon_{k+1} n_{i_{k+1}} + \dots + \varepsilon_p n_{i_p} = d - A$$

analogous to (2.5). Hence without loss of generality we may assume $k = 0$. Also, the estimate in the last line of (2.8) is trivial (since there are at most N choices for $n_{i_{k+1}} = n_{i_{k+2}}$) and the estimate in the second line follows from Lemma 3 if $k + 1 < p$ and is trivial if $k + 1 = p$. Hence it suffices to prove the estimates in the first and third lines.

In the case $s = 2$ the nontriviality of the block $(n_{i_{k+1}}, n_{i_{k+2}})$ means that either $n_{i_{k+1}} \ne n_{i_{k+2}}$ or $n_{i_{k+1}} = n_{i_{k+2}}$ but $\varepsilon_{k+1} = \varepsilon_{k+2}$. In the first case we show (assuming, as we may, $k = 0$) that there are at most $3\sqrt{N}\log N$ choices for the pair (n_{i_1}, n_{i_2}) such that $n_{i_1} \neq n_{i_2}$ and the gap n_{i_2}/n_{i_3} is large. Indeed, by the last assumption we have $n_{i_3} \le n_{i_2} \cdot c/(32p\sqrt{N}) \le n_{i_1} \cdot c/(32p\sqrt{N})$ and thus

$$|\varepsilon_3 n_{i_3} + \cdots + \varepsilon_p n_{i_p}| \le n_{i_1} \cdot c/(32\sqrt{N}).$$

On the other hand, by (2.1) and $n_{i_1} \neq n_{i_2}$ we have $n_{i_1}/n_{i_2} \ge 1 + c/\sqrt{i_2} \ge 1 + c/\sqrt{N}$ and thus

$$\begin{aligned} 2n_{i_1} &\ge |\varepsilon_1 n_{i_1} + \varepsilon_2 n_{i_2}| \ge |n_{i_1} - n_{i_2}| \\ &\ge n_{i_1}(1 - (1 + c/\sqrt{N})^{-1}) \ge n_{i_1} \cdot c/(2\sqrt{N}). \end{aligned} \tag{2.9}$$

Hence the absolute value of the left side of (2.5) lies in $[n_{i_1} \cdot c/(4\sqrt{N}), 3n_{i_1}]$ and thus (2.5) implies

$$n_{i_1} \in [|d|/3,\ 4|d|\sqrt{N}/c].$$

Thus by Lemma 1 the number of choices for n_{i_1} is at most

$$\frac{2\sqrt{N}}{c} \log \frac{12\sqrt{N}}{c} + 1 \le 3\sqrt{N} \log N.$$

With n_{i_1} chosen, there is at most one choice for n_{i_2} by Lemma 3 since the gap n_{i_2}/n_{i_3} is large. Thus in the first case of nontriviality of $(n_{i_{k+1}}, n_{i_{k+2}})$ (i.e., $n_{i_{k+1}} \neq n_{i_{k+2}}$) the estimate in the third line of (2.8) is proved. In the second case, i.e., when $n_{i_{k+1}} = n_{i_{k+2}}$ but $\varepsilon_{k+1} = \varepsilon_{k+2}$, the proof is the same except that in this case (2.9) is replaced by $|\varepsilon_1 n_{i_1} + \varepsilon_2 n_{i_2}| = 2n_{i_1}$.

We turn now to the case $s \ge 3$ in (2.8). Again, we may assume $k = 0$, i.e., we estimate the number of choices for the block $(n_{i_1}, \dots, n_{i_s})$. Fix integers $j_2, \dots, j_{s-1}$ and first estimate the number of those blocks $(n_{i_1}, \dots, n_{i_s})$ where $n_{i_\nu}/n_{i_{\nu+1}} \asymp 2^{j_\nu}$, $2 \le \nu \le s-1$. Clearly, for n_{i_1} there are at most N possibilities and given $n_{i_1}, \dots, n_{i_{\nu-1}}$, $2 \le \nu \le s-1$, Lemma 2 shows that for n_{i_ν} there are at most $(48\sqrt{N}/c) \cdot \psi(j_\nu)$ possibilities where the function $\psi(j)$, $j = 0, 1, \dots$, is defined by

$$\psi(j) = \begin{cases} 1 & \text{if } p2^{-j} \ge 1/8, \\ p2^{-j} & \text{if } p2^{-j} < 1/8. \end{cases}$$

Finally, given $n_{i_1}, \dots, n_{i_{s-1}}$, for n_{i_s} there is at most one possibility by Lemma 3 since $(n_{i_1}, \dots, n_{i_s})$ is a block, i.e., the gap $n_{i_s}/n_{i_{s+1}}$ is large. (Again, if $s = p$ then Lemma 3 does not apply but for n_{i_p} there is trivially at most one possibility if $n_{i_1}, \dots, n_{i_{p-1}}$ are given.) Thus the number of choices for $(n_{i_1}, \dots, n_{i_s})$ is at most

$$N(48\sqrt{N}/c)^{s-2} \prod_{\nu=2}^{s-1} \psi(j_\nu). \tag{2.10}$$

Summing (2.10) for $j_2, \dots, j_{s-1}$ we get an upper estimate for the number of blocks $(n_{i_1}, \dots, n_{i_s})$. Note that

$$\sum_{j=0}^{\infty} \psi(j) = \sum_{p2^{-j} \ge 1/8} 1 + \sum_{p2^{-j} < 1/8} p2^{-j} \le 2\log 8p + 1 + 1/4 \le 12 \log p \qquad (p \ge 2)$$

since the last sum is a geometric series with ratio $1/2$ and first term $\le 1/8$. Hence adding (2.10) for $j_2, \dots, j_{s-1}$ we get at most

$$N(48\sqrt{N}/c)^{s-2} \prod_{\nu=2}^{s-1} \left(\sum_{j_\nu=0}^{\infty} \psi(j_\nu) \right) \le N(48\sqrt{N}/c)^{s-2} (12 \log p)^{s-2}$$

proving the estimate in the first line of (2.8).

Lemma 5. *The number of solutions of* (2.5) *containing a block of length* 1 *or a nontrivial block of length* 2 *is at most*

$$4(1200\log p)^p N^{(p-1)/2}\log N \tag{2.11}$$

Proof. By Lemma 4 the number of solutions of (2.5) with block lengths $s_1, \dots, s_r$ $(s_i \ge 1, s_1 + \dots + s_r = p)$ is at most

$$\begin{aligned}&\prod_{\{\nu\,:\,s_\nu\ge 3\}} \frac{(600\log p)^{s_\nu}}{c^{s_\nu-2}} N^{s_\nu/2} \cdot \prod_{\{\nu\,:\,s_\nu=1\}} 1 \cdot \prod_{\substack{\{\nu\,:\,s_\nu=2 \text{ and the}\\ \nu\text{th block is trivial}\}}} N\\ &\cdot \prod_{\substack{\{\nu\,:\,s_\nu=2 \text{ and the}\\ \nu\text{th block is nontrivial}\}}} 3\sqrt{N}\log N.\end{aligned} \tag{2.12}$$

If there is a ν with $s_\nu = 1$ then $\sum_{\{\nu\,:\,s_\nu\ge 2\}} s_\nu \le p-1$ and thus using $c \ge 1$ and estimating $3\sqrt{N}\log N$ by N it follows that the expression in (2.12) is

$$\le \prod_{\{\nu\,:\,s_\nu\ge 3\}} (600\log p)^{s_\nu} N^{s_\nu/2} \cdot \prod_{\{\nu\,:\,s_\nu=2\}} N^{s_\nu/2} \le (600\log p)^p N^{(p-1)/2}$$

On the other hand, if there is a nontrivial block of length 2, i.e., the last product in (2.12) is not empty, then replacing all terms of the product by N we increase the product by at least a factor $\sqrt{N}/(3\log N)$ and thus the expression (2.12) is

$$\le \prod_{\{\nu\,:\,s_\nu\ge 3\}} (600\log p)^{s_\nu} N^{s_\nu/2} \prod_{\{\nu\,:\,s_\nu=2\}} N^{s_\nu/2} \cdot \frac{3\log N}{\sqrt{N}} \le (600\log p)^p N^{p/2}\frac{3\log N}{\sqrt{N}}.$$

Since the number of systems $(s_1, \dots, s_r)$ satisfying $1 \le r \le p$, $s_i \ge 1$, $s_1 + \dots + s_r = p$ is at most 2^p, the number of solutions of (2.5) containing a block of length 1 or a nontrivial block of length 2 is bounded by the expression in (2.1), as stated.

In the sequel, call a solution $(n_{i_1}, \dots, n_{i_p})$ of (2.5) *regular* if it contains only blocks of length ≥ 3 and trivial blocks of length 2.

Lemma 6. *Let* $L \ge 0$ *and* $p = s + t$ *where* $s \ge 0$, $t \ge 0$, *and* s, t *are even. Then the number of those regular solutions of* (2.5) *which contain* L *blocks of length* ≥ 3 *with total length* s *and* $t/2$ *trivial blocks is at most*

$$\frac{1}{(\frac{t}{2})!L!}\frac{(1200\log p)^s}{c^{s-2L}} N^{p/2}. \tag{2.13}$$

Proof. Fix integers $s_1, \dots, s_r \ge 2$ such that $p = s_1 + \dots + s_r$. It is easy to see that the number of regular solutions of (2.5) with block lengths $s_1, \dots, s_r$ is at most

$$\frac{N^r}{r!} \prod_{\{\nu\,:\,s_\nu\ge 3\}} \frac{(600\log p)^{s_\nu}}{c^{s_\nu-2}} N^{s_\nu/2-1}. \tag{2.14}$$

Indeed, fix the first element of each block. This means choosing r different elements of the sequence $n_1, \dots, n_N$, i.e., the number of choices is $\binom{N}{r} \le N^r/r!$. Once these first elements are chosen, the trivial blocks are determined and by Lemma 4 the number of choices for the remaining elements in a block of length $s_\nu \geq 3$ (assuming that all elements preceding this block are already chosen) is at most

$$\frac{(600 \log p)^{s_\nu}}{c^{s_\nu - 2}} N^{s_\nu/2-1}.$$

(The extra -1 in the exponent of N is due to the fact that in Lemma 4 the first element of the block was not fixed and we estimated the number of choices for this first element by N, while here the first element is fixed.) Hence the estimate (2.14) is verified.

Now let s, t, L be given as in the formulation of the lemma and let $M_{s,t,L}$ be the number of regular solutions of (2.5) with parameters s, t, L considered in Lemma 6. By the preceding estimates, we get an upper bound for $M_{s,t,L}$ if we add (2.14) for all systems $(s_1, \dots, s_r)$ of integers ≥ 2 satisfying $s_1 + \dots + s_r = p$ and

$$s = \sum_{\{\nu : s_\nu \geq 3\}} s_\nu, \quad t = p - s, \quad L = \sum_{\{\nu : s_\nu \geq 3\}} 1. \tag{2.15}$$

We observe first that the number of such systems $(s_1, \dots, s_r)$ is at most

$$\frac{(\frac{t}{2} + L)!}{(\frac{t}{2})! L!} 2^s. \tag{2.16}$$

Indeed, such a system can be obtained in two steps. First we choose those indices ν, $1 \leq \nu \leq r$, for which $s_\nu = 2$; since the number of such ν's is $t/2$ and $r = t/2 + L$, the number of choices is given by the fraction in (2.16). Once the indices ν with $s_\nu = 2$ are given, we have to split s in the form given by the first equation of (2.15). Clearly the number of possibilities is $\leq 2^s$, verifying the estimate (2.16).

Observe now that under the side condition (2.15) the estimate (2.14) becomes

$$\frac{1}{r!} \frac{(600 \log p)^s}{c^{s-2L}} N^{s/2-L+r} = \frac{1}{(\frac{t}{2} + L)!} \frac{(600 \log p)^s}{c^{s-2L}} N^{p/2}.$$

Multiplying this expression with (2.16) we get estimate (2.13), completing the proof of Lemma 6.

Note that in Lemmas 2–6 above we considered equation (2.5) where the signs $\varepsilon_1, \dots, \varepsilon_p$ were fixed and the indices $i_1, \dots, i_p$ were in nonincreasing order. Permuting the indices $i_1, \dots, i_p$ and choosing the signs $\varepsilon_1, \dots, \varepsilon_p$ in all possible ways we get an estimate for the number of solutions of the original equation (2.3):

Lemma 7. *Let $L \geq 0$ and $p = s + t$ where $s \geq 0$, $t \geq 0$, and s, t are even. Then the number of those regular solutions of* (2.3) *which contain L blocks of length ≥ 3 with total length s and $t/2$ trivial blocks is at most*

$$\frac{p!}{(\frac{t}{2})! L!} \frac{(2400 \log p)^s}{c^{s-2L}} N^{p/2}. \tag{2.17}$$

On the other hand, the number of nonregular solutions of (2.3) *is at most*

$$4p!(2400 \log p)^p N^{(p-1)/2} \log N. \tag{2.18}$$

Here, a solution of (2.3) is called regular if it becomes regular after arranging $i_1, \dots, i_p$ in a nonincreasing fashion. Also, the blocks in a solution of (2.3) are meant after this rearrangement.

Proof of Lemma 7. Since the number of permutations of $i_1, \dots, i_p$ is $\le p!$ ($i_1, \dots, i_p$ are not necessarily different) and the number of choices for $\varepsilon_1, \dots, \varepsilon_p$ is 2^p, estimate (2.18) is immediate from Lemma 5. To deduce (2.17) from Lemma 6 consider the regular solutions of (2.5) with block lengths $s_1, \dots, s_r$. In a block of length $s_\nu \ge 3$ there are 2^{s_ν} different choices for the signs ε_i while in a trivial block of length 2 there are only 2 choices (instead of 4) since the two signs in a trivial block are opposite. Hence if $t/2$ is the number of trivial blocks and $s = p - t$ then the total number of choices for the signs $\varepsilon_1, \dots, \varepsilon_p$ is $\le 2^s 2^{t/2}$. On the other hand, among the indices $i_1, \dots, i_p$ there are $t/2$ pairs of equal numbers and thus the number of different permutations of $i_1, \dots, i_p$ is $\le p!/2^{t/2}$. Thus passing from equation (2.5) to (2.3) means an extra factor $\le p!2^s$ in (2.13), proving (2.17).

Using Lemma 7 it is now easy to complete the proof of the Main Lemma. Let us note first that in Lemma 7 s is the total length of L blocks of length ≥ 3 and thus $s \ge 3L$; on the other hand, for $(t/2)!$ appearing in (2.17) we have

$$\left(\frac{t}{2}\right)! = \left(\frac{p}{2} - \frac{s}{2}\right)! \ge \frac{(\frac{p}{2})!}{p^{s/2}}.$$

Hence using $c \ge p^{1/2+\varepsilon}$ we get that the expression in (2.17) cannot exceed

$$\begin{aligned} \frac{p!p^{s/2}}{(\frac{p}{2})!L!} \frac{(2400 \log p)^s}{p^{s/2-L}p^{(s-2L)\varepsilon}} N^{p/2} &\le \frac{p^L}{L!} \frac{(2400 \log p)^s}{p^{s\varepsilon/3}} \frac{p!}{(p/2)!} N^{p/2} \\ &\le \frac{p^{s(1-\varepsilon)/3}}{[s/3]!} (2400 \log p)^s \frac{p!}{(p/2)!} N^{p/2} = \frac{\tau^s}{[s/3]!} \frac{p!}{(p/2)!} N^{p/2} \end{aligned} \tag{2.19}$$

where [ ] denotes integral part and

$$\tau = 2400 p^{(1-\varepsilon)/3} \log p. \tag{2.20}$$

(In the second inequality of (2.19) we used the fact that $p^n/n!$ is increasing for $0 \le n \le p-1$.) Now it is easy to see that

$$\sum_{s=0}^{\infty} \frac{(s+1)\tau^s}{[s/3]!} \le \exp(2\tau^3), \qquad \tau \ge \tau_0, \tag{2.21}$$

and thus adding the last expression in (2.19) for $0 \le L \le s/3$ and then for $s = 0, 1, 2, \dots$ we get that the number of regular solutions of (2.3) is

$$\le \exp\{K p^{1-\varepsilon} \log^3 p\} \frac{p!}{(p/2)!} N^{p/2}$$

provided that $p \ge K$ where K is an absolute constant. On the other hand, by the second statement of Lemma 7 the number of nonregular solutions of (2.3) is

$$\begin{aligned} &\le 4 \frac{p!}{(p/2)!} (2400 p \log p)^p N^{p/2-1/4} \le \frac{p!}{(p/2)!} e^{p^2} N^{p/2-1/4} \\ &\le e^{-p^2} \frac{p!}{(p/2)!} N^{p/2} \end{aligned} \tag{2.22}$$

for $N \ge K$, $K \le p \le 4 \log\log N$ since $e^{2p^2} \le N^{1/4}$. This completes the proof of the Main Lemma for $\delta = 1$.

To get the Main Lemma for $0 < \delta < 1$ let us observe that if $i_1, \dots, i_p$ are restricted to the interval $[(1-\delta)N, N]$ instead of $[0, N]$ then the estimates in the first and last lines of (2.8) have to be multiplied with δ (in fact, in the general case we have δN possibilities for the first element of each block instead of N). As a consequence, estimates (2.13) and (2.17) have to be multiplied with $\delta^{L+t/2} \le \delta^{t/2} = \delta^{p/2-s/2}$. Hence estimate (2.19) will hold in the general case if $N^{p/2}$ is replaced by $(\delta N)^{p/2}$ and $(2400 \log p)^s$ is replaced by $(2400 \log p/\sqrt{\delta})^s$, leading to a new value

$$\tau = 2400 p^{1-\varepsilon} \log^3 p/\sqrt{\delta}$$

giving the estimate (2.4). On the other hand, estimate (2.18) for the number of nonregular solutions of (2.3) remains valid for general δ and from the second condition of (2.2) (written equivalently as $N \ge \exp(\exp(p/4))$) it follows that for $N \ge N_0(\delta)$ we have

$$4p!(2400 \log p)^p \le \delta^{p/2} N^{1/4}.$$

Hence the expression in (2.18) is also bounded by (2.4), completing the proof of the Main Lemma for general δ.

In the above proof we assumed $0 < \varepsilon \le 1$, i.e., the number τ in (2.20) is large for p large. For $\varepsilon > 1$ the situation changes essentially: in this case $\tau = O(p^{-(\varepsilon-1)/6})$ is small for p large and the sum of the power series (2.21) becomes $1 + O(\tau) = 1 + O(p^{-(\varepsilon-1)/6})$. Moreover, if we exclude the trivial solutions of (2.3) (this makes a difference only for $d = 0$) then in Lemmas 6 and 7 s cannot be 0 and thus the sum in (2.21) becomes $O(\tau) = O(p^{-(\varepsilon-1)/6})$. Since the number of nonregular solutions of (2.3) is bounded by the expression in (2.22), it follows that for $\varepsilon > 1$ the factor $\exp\{Kp^{1-\varepsilon} \log^3 p/\sqrt{\delta}\}$ in (2.4) can be replaced by $1 + O(p^{-(\varepsilon-1)/6})$ or $O(p^{-(\varepsilon-1)/6})$ according as we include or exclude the trivial solutions. In other words, for $\varepsilon > 1$ and p large, almost all solutions of (2.3) with $d = 0$ are trivial.

In conclusion we note that although the gap condition (2.1) in the Main Lemma is assumed for all $1 \le j \le N-1$, our proof used it only for $(1-\delta)N \le j \le N-1$ (i.e., for the values actually appearing in (2.3)). Similarly, if we assume (2.1) for $(1-\delta)N + a \le j \le N-1$ then estimate (2.4) will be valid for the number of solutions of

$$\pm n_{i_1} \pm \cdots \pm n_{i_p} = d, \quad (1-\delta)N + a \le i_1, \dots, i_p \le N.$$

3. Proof of the theorem

Using the Main Lemma of §2, the proof our our theorem can be completed in a rather standard way. We first note the following consequence of the Main Lemma:

Lemma 8. *Let $\{n_j, 1 \le j \le N\}$ be a finite sequence of positive integers satisfying*

$$n_{j+1}/n_j \ge 1 + c/\sqrt{j}, \qquad (1-\delta)N \le j \le N-1. \tag{3.1}$$

Further let $p \ge 2$ be an even integer and assume that (2.2) *holds for some $\varepsilon > 0$. Then for $N \ge N_0(\delta)$, $p \ge K$ we have*

$$\int_0^{2\pi} \left(\sum_{(1-\delta)N \le j \le N} \cos n_j x \right)^p dx \le 2\pi 2^{-p} A_{N,p,\delta} \tag{3.2}$$

where $A_{N,p,\delta}$ is the number in (2.4). *The result remains valid if the lower limit* $(1-\delta)N$ *in* (3.1) *and in the sum in* (3.2) *is replaced by* $(1-\delta)N+a$, *provided this number is* $\le N$.

To see this, let us note that the integrand in (3.2) equals

$$2^{-p}\sum \cos(\pm n_{i_1} \pm \cdots \pm n_{i_p})x$$

where the summation is extended for all p-tuples $(i_1, \dots, i_p)$ with $(1-\delta)N \le i_1, \dots, i_p \le N$ and all choices of the signs ± 1. Since for integer n we have $\int_0^{2\pi} \cos nx\, dx = 2\pi$ or 0 according as $n = 0$ or $n \ne 0$, the result follows from the Main Lemma. (See the remark at the end of §2.)

We shall also need the following maximal inequality for trigonometric sums, proved in our paper [1]:

Lemma 9. *Let $p \ge 2$ be an even integer and $f \in L^p(0, 2\pi)$ an even function with nonnegative Fourier coefficients. Let $s_n(f)$ denote the nth partial sum of the Fourier series of f. Then*

$$\int_0^{2\pi} \left(\sup_{k\ge 1} |s_{2^k}(f)|\right)^p dx \le A^p \int_0^{2\pi} |f|^p\, dx$$

where A is an absolute constant.

Now let (n_k) be a sequence of positive integers satisfying (1.5) with $\alpha = 1/2+\varepsilon$, $\varepsilon > 0$. Let $N \ge 1$, $\lambda > 1$ and set

$$p = 2[\log\log N], \qquad c = (\log\log\sqrt{N})^{1/2+\varepsilon}.$$

Then we have

$$n_{j+1}/n_j \ge 1 + c/\sqrt{j}, \qquad \sqrt{N} \le j \le N-1;$$

further (2.2) holds for $N \ge N_0$ with ε replaced by $\varepsilon/2$. Hence applying Lemma 8 with $\delta = 1$, $a = \sqrt{N}$ and using $p!/(p/2)! \sim \sqrt{2}(2p/e)^{p/2}$ and the Markov inequality, we get

$$\begin{aligned}
&\lambda\left\{0 \le x \le 2\pi \colon \left|\sum_{\sqrt{N}\le k\le N} \cos n_k x\right| > (\lambda N \log\log N)^{1/2}\right\}\\
&\le (\lambda N \log\log N)^{-p/2} 2\pi 2^{-p} \exp\{Kp^{1-\varepsilon/4}\} \frac{p!}{(p/2)!} N^{p/2}\\
&\le \lambda^{-p/2}(p/2)^{-p/2} 2\pi 2^{-p} \exp\{Kp^{1-\varepsilon/4}\} 2(2p/e)^{p/2}\\
&= 4\pi(\lambda e)^{-p/2} \exp\{Kp^{1-\varepsilon/4}\}\\
&\le 4\pi \exp\{-(1+\log\lambda)[\log\log N] + 2K[\log\log N]^{1-\varepsilon/4}\}\\
&\le \exp\{-(1+\tau)\log\log N\}
\end{aligned}$$

for $N \ge N_0$ where $\tau = \frac{1}{2}\log\lambda > 0$. Thus setting $S_N = \sum_{k\le N} \cos n_k x$ and choosing $N_k = [a^k]$, $1 < a < 2$, it follows from the above estimates and the Borel-Cantelli lemma that

$$\limsup_{k\to\infty} \frac{|S_{N_k} - S_{[N_k^{1/2}]}|}{(N_k \log\log N_k)^{1/2}} \le \sqrt{\lambda} \quad \text{a.e.}$$

Clearly $|S_{[N_k^{1/2}]}| \le N_k^{1/2}$ and thus the last relation implies

$$\limsup_{k\to\infty} \frac{|S_{N_k}|}{(N_k \log\log N_k)^{1/2}} \le \sqrt{\lambda} \quad \text{a.e.}$$

Since λ and a can be chosen arbitrarily close to 1, our theorem will follow if we show that setting

$$M_k = \max_{N_k \le j \le N_{k+1}} |S_j - S_{N_k}|$$

we have

$$\limsup_{k\to\infty} \frac{M_k}{(N_k \log\log N_k)^{1/2}} \le 8A(a-1)^{1/2} \quad \text{a.e.} \tag{3.3}$$

where A is the absolute constant appearing in Lemma 9.

To prove (3.3) set

$$Z_k = \sum_{j=N_k+1}^{N_{k+1}} \cos n_j x\,, \quad p(k) = \max\{i\colon n_i \le 2^k\}\,, \quad H = \{p(1), p(2), \dots\}\,.$$

Clearly

$$M_k \le \max_{\substack{N_k \le j \le N_{k+1}\\ j\in H}} |S_j - S_{N_k}| + \max_{\{i\,:\,p(i)\le N_{k+1}\}} |p(i+1) - p(i)| := J_1 + J_2\,. \tag{3.4}$$

Using (1.5) with $\alpha > 1/2$ we get for $k \ge k_0$

$$\begin{aligned} 2 \ge n_{p(k+1)}/n_{p(k)+1} &\ge \prod_{m=p(k)+1}^{p(k+1)-1} (1 + 1/\sqrt{m}) \\ &\ge 1 + \sum_{m=p(k)+1}^{p(k+1)-1} 1/\sqrt{m} \ge 1 + (p(k+1) - p(k) - 1)/\sqrt{p(k+1)} \end{aligned}$$

whence it follows that $p(k+1)/p(k) \to 1$ and

$$p(k+1) - p(k) = O(p(k)^{1/2})\,.$$

Thus for J_2 in (3.4) we have

$$|J_2| = O(N_{k+1}^{1/2}) = o(N_k \log\log N_k)^{1/2}\,. \tag{3.5}$$

On the other hand, choosing $p = 2[\log\log N_k]$ and applying Lemma 9 with $f = Z_k$ we get

$$\int_0^{2\pi} |J_1|^p\, dx \le A^p \int_0^{2\pi} |Z_k|^p\, dx \tag{3.6}$$

where A is an absolute constant. Using (1.5) with $\alpha > 1/2$ it follows that relation (3.1) holds with $N = N_{k+1}$, $c = (\log\log N_k)^{1/2+\varepsilon}$, $\delta = 1 - 1/a$ and (2.2) is also valid for $k \ge k_0$, with ε replaced by $\varepsilon/2$. Hence by Lemma 8 the integral on the right-hand side of (3.6) is at most

$$2\pi 2^{-p} \exp\left\{ \frac{Kp^{1-\varepsilon/4}}{\sqrt{1 - 1/a}} \right\} \frac{p!}{(p/2)!} 2(N_{k+1} - N_k)^{p/2}\,.$$

Hence (3.6) and the Markov inequality imply for $k \geq k_0$

$$\begin{aligned}
&\lambda\{0 \leq x \leq 2\pi : |J_1| \geq 4A(N_{k+1} - N_k)^{1/2}(\log\log N_k)^{1/2}\} \\
&\quad \leq (4A)^{-p}(N_{k+1} - N_k)^{-p/2}(\log\log N_k)^{-p/2} A^p 2\pi 2^{-p} \\
&\qquad \cdot \exp\left\{\frac{Kp^{1-\varepsilon/4}}{\sqrt{1-1/a}}\right\} \frac{p!}{(p/2)!} 2(N_{k+1} - N_k)^{p/2} \\
&\quad \leq 4^{-p}(p/2)^{-p/2} 2\pi 2^{-p} \exp(p/2) 4 (2p/e)^{p/2} \leq 8\pi 4^{-p} \leq k^{-3/2}.
\end{aligned}$$

Hence by the Borel-Cantelli lemma and $N_{k+1} - N_k \sim (a-1)N_k$ we have

$$|J_1| \leq 8A(a-1)^{1/2}(N_k \log\log N_k)^{1/2} \quad \text{a.e. for } k \geq k_0. \tag{3.7}$$

Now (3.4), (3.5), and (3.7) imply (3.3), completing the proof of the theorem.

4. Remarks

In this section we construct an example showing that for small enough $\varepsilon > 0$ the subexponential factor

$$\exp\left\{\frac{Kp^{1-\varepsilon}\log^3 p}{\sqrt{\delta}}\right\}$$

in (2.4) cannot be replaced by $O(1)$ or even by $o(p^{2-2\varepsilon})$. Let $a_k = (k!)^2$, $m_k = [k/(\log\log k)^{2\alpha}]$ $(k \geq 3)$, $m_1 = m_2 = 0$, and $M_k = \sum_{i=1}^k m_i$ for some $\alpha > 1/2$. Let $I_k = \{a_k, 2a_k, \dots, m_k a_k\}$; clearly the sets I_k, $k = 1, 2, \dots$, are disjoint. Define the sequence (n_k) by $(n_k) = \bigcup_{j=1}^\infty I_j$. It is easily seen that

$$\frac{n_{j+1}}{n_j} \geq 1 + \frac{(\log\log j)^\alpha}{2\sqrt{j}}, \qquad j \geq j_0. \tag{4.1}$$

Indeed, if $M_{k-1} < j < M_k$ then setting $i = j - M_{k-1}$ and using

$$M_k \sim \frac{k^2}{2(\log\log k)^{2\alpha}} \qquad (k \to \infty)$$

and $M_k/M_{k-1} \to 1$ we get

$$\begin{aligned}
\frac{n_{j+1}}{n_j} &= 1 + \frac{1}{i} \geq 1 + \frac{1}{m_k} \geq 1 + \frac{(\log\log k)^{2\alpha}}{k} \geq 1 + \frac{(\log\log M_k)^\alpha}{2\sqrt{M_{k-1}}} \\
&\geq 1 + \frac{(\log\log j)^\alpha}{2\sqrt{j}}, \qquad j \geq j_0.
\end{aligned}$$

On the other hand, if $j = M_k$ then $n_{j+1}/n_j = a_{k+1}/(m_k a_k) \geq 2$; i.e., (4.1) holds in this case too. Let $N = M_k$, $0 < \delta \leq 1$; we show that the number of nontrivial solutions of

$$\pm n_{i_1} \pm \cdots \pm n_{i_p} = 0, \quad (1-\delta)N \leq i_1, \dots, i_p \leq N, \tag{4.2}$$

is at least

$$\text{const} \cdot p^{2-2\alpha} \frac{p!}{(p/2)!} (\delta N)^{p/2} \tag{4.3}$$

for any even number p satisfying $\log\log N \leq p \leq 4\log\log N$. We consider again the case $\delta = 1$; for general δ the proof is similar. Let us call a solution

$(n_{i_1}, \dots, n_{i_p})$, $1 \le i_1 \le i_2 \le \cdots \le i_p \le N$ of (4.2) (where we take $\delta = 1$) *almost trivial* if the n_{i_ν}'s in (4.2) are pairwise equal with opposite signs except one 4-tuple $n_{i_{s_1}} < n_{i_{s_2}} < n_{i_{s_3}} < n_{i_{s_4}}$ belonging to the set I_l for some $1 \le l \le k$ and satisfying

$$n_{i_{s_1}} + n_{i_{s_2}} + n_{i_{s_3}} - n_{i_{s_4}} = 0. \tag{4.4}$$

Moreover, we stipulate that the elements of different pairs are different and they all differ from the elements of the 4-tuple $(n_{i_{s_1}}, n_{i_{s_2}}, n_{i_{s_3}}, n_{i_{s_4}})$. Since $I_l = \{ja_l, 1 \le j \le m_l\}$, clearly for each triplet $(n_{i_{s_1}}, n_{i_{s_2}}, n_{i_{s_3}})$ lying in I_l and satisfying $n_{i_{s_1}} < n_{i_{s_2}} < n_{i_{s_3}} < a_l m_l/3$ there exists a unique $n_{i_{s_4}}$ in I_l such that (4.4) holds. Thus the number of 4-tuples $n_{i_{s_1}} < n_{i_{s_2}} < n_{i_{s_3}} < n_{i_{s_4}}$ lying in I_l and satisfying (4.4) is at least $\text{const}\cdot m_l^3$. Hence in an almost trivial solution the number of choices for the exceptional 4-tuple $(n_{i_{s_1}}, n_{i_{s_2}}, n_{i_{s_3}}, n_{i_{s_4}})$ is at least

$$\text{const}\cdot\sum_{l=1}^{k} m_l^3 \sim \text{const}\cdot\frac{k^4}{(\log\log k)^{6\alpha}} \sim \text{const}\cdot\frac{N^2}{(\log\log N)^{2\alpha}} \ge \text{const}\cdot\frac{N^2}{p^{2\alpha}}.$$

Once $(n_{i_{s_1}}, n_{i_{s_2}}, n_{i_{s_3}}, n_{i_{s_4}})$ is fixed, the number of choices for the remaining $(p-4)/2$ pairs clearly equals

$$\binom{N-4}{(p-4)/2} \ge \frac{(N-p)^{p/2-2}}{(p/2-2)!} \ge \frac{p^2}{8}\frac{N^{p/2-2}}{(p/2)!}$$

for $N \ge N_0$ since $p \le 4\log\log N$ implies $(N-p)^{p/2-2} \sim N^{p/2-2}$. Thus the total number of choices for the almost trivial solution $(n_{i_1}, \dots, n_{i_p})$, $1 \le i_1 \le i_2 \le \cdots \le i_p \le N$ is

$$\ge \text{const}\cdot p^{2-2\alpha}\frac{N^{p/2}}{(p/2)!}.$$

Since the number of choices for the signs in (4.2) is at least $2^{p/2-2}$ (in each pair (n_{i_q}, n_{i_r}) with $n_{i_q} = n_{i_r}$ there are two choices of the signs ε_{i_q}, ε_{i_r} such that $\varepsilon_{i_q} = -\varepsilon_{i_r}$ and since the number of permutations of $(n_{i_1}, \dots, n_{i_p})$ is $p!/2^{p/2-2}$, our estimate (4.3) is proved.

In view of the proof of Lemma 8, estimate (4.3) with $\delta = 1$ implies that

$$\int_0^{2\pi}\left(\sum_{k\le N}\cos n_k x\right)^p dx \ge \text{const}\cdot p^{2-2\alpha}\frac{p!}{2^p(p/2)!}N^{p/2} \tag{4.5}$$

for $\log\log N \le p \le 4\log\log N$. On the other hand, observe that if ξ_k, $k = 1, 2, \dots$, are independent normal random variables having the same mean and variance as $\cos n_k x$ then

$$E\left(\sum_{k\le N}\xi_k\right)^p = \frac{p!}{2^p(p/2)!}N^{p/2}, \qquad p = 2, 4, \dots.$$

Thus (4.5) expresses the surprising fact that under (1.5), $\alpha > 1/2$ the asymptotic behavior of the high moments of $\sum_{k\le N}\cos n_k x$ can be different from that of sums of i.i.d. r.v.'s even though $(\cos n_k x)$ satisfies the central limit theorem (1.2) and the law of the iterated logarithm (1.6). This is another example of the unusual properties of $(\cos n_k x)$ under (1.5), discussed in §1. On the other hand, in §2 we proved that for $\varepsilon > 1$ the number of nontrivial solutions of (2.3) is $O(p^{-(\varepsilon-1)/6})$ and thus under (1.5) with $\alpha > 3/2$ we have

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k\le N}\cos n_k x\right)^p \sim \frac{p!}{2^p(p/2)!}N^{p/2}$$

as $N \to \infty$, uniformly for $2 \le p \le 4\log\log N$. Thus for $\alpha > 3/2$ the above pathological moment behavior cannot occur.

REFERENCES

1. I. Berkes, *Critical LIL behavior of the trigonometric system*, Trans. Amer. Math. Soc. **338** (1993), 553–585.
2. P. Erdős, *On trigonometric sums with gaps*, Magyar Tud. Akad. Mat. Kut. Int. Közl. **7** (1962), 37–42.
3. P. Erdős and I. S. Gál, *On the law of the iterated logarithm*, Nederl. Akad. Wetensch. Proc. Ser. A **58** (1955), 65–84.
4. W. Feller, *The general form of the so-called law of the iterated logarithm*, Trans. Amer. Math. Soc. **54** (1943), 373–401.
5. R. Salem and A. Zygmund, *On lacunary trigonometric series*, Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 333–338.
6. ——, *La loi du logarithme itéré pour les séries trigonométriques lacunaires*, Bull. Sci. Math. (2) **74** (1950), 209–224.
7. S. Takahashi, *On the law of the iterated logarithm for lacunary trigonometric series*, Tôhoku Math. J. **24** (1972), 319–329.
8. ——, *On the law of the iterated logarithm for lacunary trigonometric series*. II, Tôhoku Math. J. **27** (1975), 391–403.

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